MATHEMATICS Proceedings A 83 (4), December 12, 1980

Asymptotics of a nonlinear delay differential equation

by J.J.A.M. Brands

Dept. of Mathematics, Eindhoven University of Technology, the Netherlands

Communicated by Prof. N.G. de Bruijn at the meeting of December 22, 1979

ABSTRACT

Sufficient conditions will be given for the existence of the limit of $f(x)$ for $x \to \infty$ if f is a solution of $w(x)f'(x) = g(f(x-1) - f(x))$.

1. INTRODUCTION

In this note we consider the delay differential equation

$$(1.1) \qquad w(x)f'(x) = g(f(x-1) - f(x)), \qquad x \geq 1,$$

where

$$(1.2) \qquad \left\{ \begin{array}{l} w \text{ is a positive continuous function on } [1, \infty), \text{ and } g \text{ is an odd real} \\ \text{valued continuously differentiable function on } \mathbb{R} \text{ such that} \\ g(x) > 0 \text{ if } x > 0. \end{array} \right.$$

We say that f is a solution of (1.1) if f is a real valued function, continuous on $[0, \infty)$, differentiable on $[1, \infty)$, satisfying (1.1) on $[1, \infty)$. Every given continuous function φ on $[0, 1]$ can be extended to a solution of (1.1), and this extension is unique. We just solve (1.1) as an ordinary differential equation on $[1, 2]$ with initial value $f(1) = \varphi(1)$, and repeat this process for the intervals $[n, n+1]$, $n = 2, 3, \ldots$.

N.G. de Bruijn (1949, 1950) and J.J.A.M. Brands (1972) treated the linear case of equation (1.1) (i.e. $g(x) = x$) for a large class of functions w (containing e.g. all functions $w(x) = x^{-\alpha}$, α real). Among other things, they proved that under some conditions for w (which for the specialization $w(x) = x^{-\alpha}$ reduce to the condition $\alpha \leq \frac{1}{2}$) every solution has a limit.

J.L. Kaplan, M. Sorg and J.A. Yorke (1979) proved for a typical autonomous equation (with a so-called order relation as righthand side) that every solution has a limit. The autonomous case of equations (1.1) (i.e. $w(x) =$ constant > 0) is a specialization of their equation.

2. RESULTS

We present the results of this note in three theorems all of which have the following form:

THEOREM 2.k. *If w and g satisfy condition* (1.2) *and, in addition, the condition* (2.k), *then for all solutions f of* (1.1) $\lim_{x\to\infty} f(x)$ *exists.*

Theorems 2.1, 2.2, and 2.3 are obtained by substitution of $k = 1$, 2, and 3. Thus, we obtain theorem 2.1 if the additional condition (2.1) is satisfied, etc. The additional conditions are:

$$\liminf_{x\to\infty} \int_x^{x+1} (w(t))^{-1} dt = 0. \tag{2.1}$$

$$\left\{\begin{array}{l} w \textit{ is continuously differentiable on } [1, \infty), \\ w \textit{ has a positive lower bound, say } w(x) \geq L > 0, \\ \int_2^x (w'(t))^2 dt = o(x)\ (x \to \infty), \\ g' \textit{ is positive and nondecreasing on } (0, \infty). \end{array}\right. \tag{2.2}$$

$$\left\{\begin{array}{l} w \textit{ is continuously differentiable on } [1, \infty) \textit{ with } w' \textit{ nondecreasing}, \\ w(x) \to 0\ (x \to \infty),\ w'(x) = o(w(x))\ (x \to \infty), \\ g' \textit{ is positive and nondecreasing, } g/g' \textit{ is nondecreasing on some} \\ \textit{interval } [0, a] \textit{ with } a > 0 \textit{ (or, equivalently, } xh'(x) \textit{ is nondecreasing} \\ \textit{on some interval } [0, b] \textit{ with } b > 0), \\ \int_2^\infty [w(x)h'(w(x))]^2 dx = \infty, \textit{ where } h \textit{ is the inverse function of } g. \end{array}\right. \tag{2.3}$$

The very simple proof of theorem 2.1 is given in section 3 as an application of lemma 3.1. The proofs of theorems 2.2 and 2.3 are presented in sections 4 and 5, and obtained by adaptations of the one in Brands (1972).

In order to give an idea of that method we present a short demonstration for a special case, viz.

$$x^{-\frac{1}{2}} f'(x) + f(x) = f(x-1).$$

Squaring, integrating from 1 to n, and integrating $2\int_1^n x^{-\frac{1}{2}} f(x) f'(x) dx$ by parts, we get

$$\int_1^n x^{-1}(f'(x))^2 dx = \int_0^1 (f(x))^2 - \int_{n-1}^n (f(x))^2 dx + (f(1))^2 -$$
$$- n^{-\frac{1}{2}}(f(n))^2 - \tfrac{1}{2}\int_1^n x^{-3/2}(f(x))^2 dx.$$

We conclude that $\int_1^\infty x^{-1}(f'(x))^2dx<\infty$, from which we can derive that $\lim_{x\to\infty} f(x)$ exists.

This method can be modified so as to be applicable, not to (1.1) itself, but to the equation obtained by differentiation.

REMARK. All conditions on w in (2.2) and (2.3) can be weakened by requiring these conditions on a sub-interval $[1+b,\infty)$ only. This is easy to see by application of theorem 2.2 or 2.3 to $f(x+b)$.

EXAMPLE. Consider the equation

$$x^{-\alpha}f'(x)=|f(x-1)-f(x)|^{\beta}\operatorname{sgn}(f(x-1)-f(x)),$$

with $\beta\geq 1$. If $\alpha\leq\beta/2$ then (1.2) and (2.3) are satisfied, hence $\lim_{x\to\infty} f(x)$ exists for a solution f, which, for $\beta=1$ and $w(x)=x^{-\alpha}$, is in agreement with results of N.G. de Bruijn (1950) and J.J.A.M. Brands (1972). If $\beta>1$ and $\alpha>\beta/2$ then the asymptotic behaviour is not known. (The case $\beta=1$, $\alpha>1/2$ is treated in N.G. de Bruijn (1949)).

3. PRELIMINARIES

Let conditions (1.2) be satisfied and let f be a solution of (1.1). We define functions M, m, δ and A by

$$M(x):=\max\{f(t)\,|\,x\leq t\leq x+1\},\ m(x):=\min\{f(t)\,|\,x\leq t\leq x+1\},$$
$$A(x):=\max\{|f'(t)|\ \ |\,x\leq t\leq x+1\},\ \delta(x)=M(x)-m(x),\ \text{for } x\geq 1.$$

LEMMA 3.1. *The solution f is bounded. Moreover, the functions M and $-m$ are nonincreasing. Furthermore, f' has at least one zero in every interval $[x,x+1]$, $x\geq 1$. If w is continuously differentiable, then f'' exists and is continuous on $[2,\infty)$, and*

$$\int_x^{x+1}|f''(s)|\,ds\geq A(x)\ \textit{for}\ x\geq 2.$$

Also

$$A(x)\geq\int_x^{x+1}|f'(s)|\,ds\geq\delta(x)\ \textit{for}\ x\geq 1.$$

PROOF OF LEMMA 3.1. The proof of the statement about M and $-m$ is obtained by obvious modifications from the one in Brands (1972). The other statements in lemma 3.1. are even simpler.

An obvious consequence of lemma 3.1 is

COROLLARY 3.1. *If $\lim_{x\to\infty}\delta(x)=0$ then $\lim_{x\to\infty} f(x)$ exists.*

PROOF OF THEOREM 2.1. Since $g(f(x-1)-f(x))$ is bounded, we have

$$\delta(x) \leq \int_x^{x+1} |f'(s)|\,ds = O\left(\int_x^{x+1} (w(t))^{-1}dt\right) \qquad (x \geq 1).$$

Since δ is monotonic, the theorem follows.

We mention several simple statements about the inverse function h of g if g satisfies the extra condition that g' is positive and nondecreasing. The proofs are easy and therefore ommitted.

h is an odd continuous function on $\mathbb{R}$, positive on $(0,\infty)$, continuously differentiable on $\mathbb{R}\setminus\{0\}$. The derivative h' is positive on $\mathbb{R}\setminus\{0\}$, nonincreasing on $(0,\infty)$, and $h'(x)=[g'(h(x))]^{-1}$ if $x \neq 0$. If $g'(0)\neq 0$, then the exclusion of $x=0$ in the foregoing statements about h can be ommitted.

LEMMA 3.2. *Suppose that, in addition to* (1.2), *the following conditions are fulfilled: g' is positive and nondecreasing on $(0,\infty)$, w is continuously differentiable on $[1,\infty)$, $w(x)\to 0$ $(x\to\infty)$, w' is negative and nondecreasing, and $w(x)/w(x+1)$ has an upper bound on $[1,\infty)$, say W. Then f' is bounded.*

PROOF OF LEMMA 3.2. For a constant solution f (i.e. $f(x)=$ constant on $[0,\infty)$) lemma 3.2. is obviously true. From now on we suppose that f is a non-constant solution. Then $A(n)>0$ $(n\geq 1)$, and expressions $h'(w(x_n)A(n))$, appearing in the sequel of this proof, are defined. For every $n\geq 2$ there is a number $x_n\in[n,n+1]$ such that $A(n)=|f'(x_n)|$. If $x_n=n$ then $A(n)\leq A(n-1)$. If $n<x_n\leq n+1$ then $f''(x_n)\ \mathrm{sgn}\ (f'(x_n))\geq 0$. Differentiating (1.1) we get

$$\text{(3.1)} \qquad w(x)f''(x)+w'(x)f'(x)=(f'(x-1)-f'(x))g'(h(w(x)f'(x))) \qquad (x\geq 2).$$

It follows that

$$[f'(x_n-1)\ \mathrm{sgn}\ (f'(x_n))-A(n)]g'(h(w(x_n)A(n)))-w'(x_nA(n))\geq 0 \ \ (n\geq 2).$$

Hence we always have

$$\text{(3.2)} \qquad [1+h'(w(x_n)A(n))w'(x_n)]A(n)\leq A(n-1) \qquad (n\geq 2).$$

Consider the function $\Psi\colon [0,\infty)\to\mathbb{R}$, defined by

$$\Psi(t)=[1-\alpha h'(\beta t)]t,$$

where α and β are positive numbers. Since $xh'(x)\leq h(x)$ $(x>0)$ we have that $\alpha t h'(\beta t)\leq \alpha\beta^{-1}h(\beta t)$. It follows that $\Psi(t)\to 0$ if $t\to 0$. From the conditions on g we know that $h'(\beta t)$ tends to a finite limit $h'(\infty)\geq 0$ if $t\to\infty$. If $\alpha h'(\infty)<1$ then clearly $\Psi(t)\to\infty$ if $t\to\infty$, and it follows that, given a positive number γ, there is a largest number t such that $\Psi(t)=\gamma$. Hence it is possible to define a sequence B_n, $n\geq n_0$, as follows: $B_{n_0}=A_{n_0}$, $B_n[1+w'(x_n)h'(w(x_n)B_n)]=B_{n-1}$ $(n\geq n_0)$, where n_0 is such that $w'(n_0)h'(\infty)>-1$, and where it is meant that, for $n>n_0$, B_n is the largest number satisfying the equality. Trivially we have that $A(n)\leq B_n<B_{n+1}$ for $n\geq n_0$. Hence

(3.3) $\qquad B_n[1+w'(x_n)h'(w(x_n)B_{n_0})]\le B_{n-1} \qquad (n>n_0),$

where we have used the monotonicity of h'. Clearly, B_n is bounded if

$$\sum_{n=n_0}^{\infty} |w'(x_n)h'(w(x_n)B_{n_0})| <\infty.$$

We have, for $n-1\le x\le n$, $n\ge n_0+1$,

$$|w'(x_n)h'(w(x_n)B_{n_0})|\le -w'(n)h'(w(n+1)B_{n_0})\le$$
$$\le -w'(n)h'(B_{n_0}W^{-2}w(n-1))\le -w'(x)h'(B_{n_0}W^{-2}w(x)).$$

Hence, for $N>n_0$,

$$\sum_{n=n_0+1}^{N} |w'(x_n)h'(w(x_n)B_{n_0})| \le -\int_{n_0}^{N} w'(x)h'(w(x)B_{n_0}W^{-2})dx\le$$
$$\le B_{n_0}^{-1}W^2h(B_{n_0}W^{-2}w(n_0)).$$

This completes the proof of lemma 3.2.

LEMMA 3.3. *Suppose that, in addition to* (1.2), *w is continuously differentiable on* $[1,\infty)$. *Then*

$$\int_2^y [f'(x-1)-f'(x)]^2dx\le C-2\int_2^y G(f(x-1)-f(x))(w(x))^{-2}w'(x)dx \ (y\ge 2),$$

where C is a positive constant and $G(t):=\int_0^t g(s)ds$.

PROOF OF LEMMA 3.3. Putting $u(x):=f(x-1)-f(x)$, we derive from (1.1) that

(3.4) $\qquad u'(x)+(w(x))^{-1}g(u(x))=(w(x-1))^{-1}g(u(x-1)) \qquad (x\ge 2).$

Squaring both sides of (3.4), integrating from 2 to y, $y>2$, integrating

$$2\int_2^y (w(x))^{-1}g(u(x))u'(x)dx$$

by parts and rearranging terms, we find

$$(3.5)\quad \begin{cases} \int_2^y (u'(x))^2dx=\int_1^2 (f'(x))^2dx+2(w(2))^{-1}G(u(2))- \\ \quad -2\int_2^y G(u(x))(w(x))^{-2}w'(x)dx-\int_{y-1}^y (f'(x))^2dx-2(w(y))^{-1}G(u(y)). \end{cases}$$

Since $G(t)\ge 0$ for all $t\in\mathbb{R}$ the lemma follows.

4. PROOF OF THEOREM 2.2

In the sequel symbols C_1, C_2, etc. denote properly chosen constants.

Since $g(f(x-1)-f(x))$ is bounded and $w(x)\ge L>0$ on $[1,\infty)$ it follows from

(1.1) that f' is bounded on $[1, \infty)$. Using the boundedness of f, f' and wf' we infer from (3.1).

$$(4.1) \qquad w(x)|f''(x)| \leq C_1 |f'(x-1) - f'(x)| + C_2 |w'(x)| \qquad (x \geq 2).$$

Squaring both sides of (4.1), using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, integrating from 2 to n, $n > 2$, we obtain

$$(4.2) \qquad \int_2^n [w(x)f''(x)]^2 dx \leq C_3 + C_4 \int_2^n (w'(x))^2 dx + C_5 \int_2^n [f'(x-1) - f'(x)]^2 dx.$$

By Lemma 3.3, using the boundedness of f and $1/w$, and using the Schwarz's inequality, we have

$$\int_2^n [f'(x-1) - f'(x)]^2 dx \leq C_6 + C_7 \int_2^n |w'(x)| dx \leq C_6 + C_7 n^{\frac{1}{2}} [\int_2^n |w'(x)|^2 dx]^{\frac{1}{2}}.$$

Moreover, by lemma 3.1 and Schwarz's inequality

$$\int_2^n [w(x)f''(x)]^2 dx \geq L^2(n-2)(\delta(n))^2.$$

Since $\int_2^n |w'(x)|^2 dx = o(n)$ $(n \to \infty)$ we can conclude that $\delta(n) = o(1)$ $(n \to \infty)$.

5. PROOF OF THEOREM 2.3

From $w'(x) = o(w(x))$ $(x \to \infty)$ it follows that $w(x)/w(x+1)$ has an upper bound W. Hence lemma 3.2 is applicable. Let $K > 1$ be an upper bound of $|f'|$. We infer from (3.1)

$$(5.1) \qquad w(x)h'(Kw(x))|f''(x)| \leq |f'(x-1) - f'(x)| + K|w'(x)|h'(Kw(x)).$$

Squaring both sides of (5.1), using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, integrating from 2 to n, $n > 2$, we obtain

$$(5.2) \qquad \left\{ \begin{aligned} &\int_2^n [w(x)h'(Kw(x))f''(x)]^2 dx \leq 2 \int_2^n [f'(x-1) - f'(x)]^2 dx + \\ &\qquad\qquad + 2K^2 \int_2^n [w'(x)h(Kw(x))]^2 dx. \end{aligned} \right.$$

Application of lemma 3.3, using the boundedness of f', and integrating by parts, using $\frac{d}{dt} G(h(t)) = th'(t)$, we can write

$$\int_2^n [f'(x-1) - f'(x)]^2 dx \leq C_8 - 2 \int_2^n G(h(Kw(x)))(w(x))^{-2} w'(x) dx =$$

$$= C_9 + 2(w(n))^{-1} G(h(Kw(n))) - 2K^2 \int_2^n h'(Kw(x)) w'(x) dx =$$

$$= C_{10} + 2(w(n))^{-1} G(h(Kw(n))) - 2Kh(Kw(n)).$$

From $G(x) \le xg(x)$ it follows that $(w(n))^{-1}G(h(Kw(n))) \ge Kh(Kw(n))$. We conclude that $\int_2^n [f'(x-1)-f'(x)]^2dx$ is bounded. Since h' is nondecreasing we have that

$$\int_2^n [w'(x)h'(Kw(x))]^2dx \le \int_2^n [w'(x)h'(w(x))]^2dx.$$

Hence, an upper bound for the right hand side of (5.2) is

$$C_{11} + 2K^2 \int_2^n [w'(x)h'(w(x))]^2dx.$$

We can find a lower bound for the left hand side of (5.2) as follows: For k_0 sufficiently large, the function $yw(x)h'(yw(x))$ is nonincreasing in x for $x \ge k_0$, and nondecreasing in y for $0<y\le K$. Let $k_0 \le k \le n-1$. Then, using Schwarz's inequality and lemma 3.1 we have

$$I_k := \int_k^{k+1} [w(x)h'(Kw(x))f''(x)]^2dx \ge K^{-2} \int_k^{k+1} [w(x)h'(w(x))f''(x)]^2dx \ge$$

$$\ge K^{-2}[w(k+1)h'(w(k+1))]^2(\delta(k))^2 \ge K^{-2}(\delta(n))^2 \int_{k+1}^{k+2} [w(x)h'(w(x))]^2dx.$$

Hence

$$\int_2^n [w(x)h'(Kw(x))f''(x)]^2dx \ge K^{-2}(\delta(n))^2 \int_{k_0+1}^n [w(x)h'(w(x))]^2dx.$$

We conclude that

$$(\delta(n))^2 \int_{k_0+1}^n [w(x)h'(w(x))]^2dx \le C_{12} + 2K^4 \int_2^n [w'(x)h'(w(x))]^2dx.$$

Using $w'(x) = o(w(x))$ $(x\to\infty)$ we easily infer that $\delta(n) = o(1)$ $(n\to\infty)$.

REFERENCES

Bruijn, N.G. de – The asymptotically periodic behaviour of some linear functional equations, Amer. J. Math. **71**, 313-330 (1949).

Bruijn, N.G. de – On some linear functional equations, Publ. Math. Debrecen **1**, 129-134 (1950).

Brands, J.J.A.M. – The asymptotic behaviour of the solutions of certain difference differential equations, Nederl. Akad. Wetensch. Proc. Ser **A 75** = Indag. Math. **34**, 73-81 (1972).

Kaplan, J.L., M. Sorg and J.A. Yorke – Solutions of $x'(t) = f(x(t), x(t-L))$ have limits when f is an order relation, Nonlinear Anal., Theory, Methods Appl. **3**, 53-58 (1979).